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On $\mathcal{N} = 1$ exact superpotentials from $U(N)$ matrix models

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Abstract

In this letter we compute the exact effective superpotential of $\mathcal{N} = 1$ $U(N)$ supersymmetric gauge theories with N_f fundamental flavors and an *arbitrary* tree-level polynomial superpotential for the adjoint Higgs field. We use the matrix model approach in the maximally confining phase. When restricted to the case of a tree-level even polynomial superpotential, our computation reproduces the known result of the $SU(N)$ theory.

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In a series of papers [1, 2] Dijkgraaf and Vafa argued that for a wide class of $\mathcal{N} = 1$ $U(N)$ supersymmetric gauge theories the effective superpotential, thought as a function of the chiral glueball superfield S, could be computed by means of a simple matrix model whose action is the tree-level superpotential of the gauge theory. Their proposal was the result of a detailed study of the various dualities between $U(N)$ supersymmetric gauge theories, B-model topological strings and matrix models in the large N limit [3, 4, 5, 6, 7]. This kind of limit was actually an old idea due to 't Hooft [8] who had shown that perturbation theory in the large N limit singles out only the planar Feynman diagrams. What Diikgraaf and Vafa discovered was that a leading-order perturbative calculation via a matrix model could capture completely an exact quantity in the corresponding gauge theory, namely the effective superpotential which describes the effects of gaugino condensation. In particular they showed that for a theory with a matter field in the adjoint representation of the gauge group the effective superpotential is always of the form:

$$W_{eff}(S) = N\frac{\partial \mathcal{F}_{\chi=2}(S)}{\partial S} \tag{1}$$

where $\mathcal{F}_{\chi=2}$ represents the planar free energy from diagrams with the topology of the sphere. Then this result was obtained directly using a perturbative field theory approach [9], and through the analysis of the generalized Konishi anomaly [10]. Many extensions of this idea have been studied, with the aim to include also quark fields in the (anti)fundamental representation [11, 12, 13, 14, 15, 16, 17, 18]. In particular in [13] it was realized that such a generalization could be achieved by taking into account the contributions from planar diagrams with the topology of the disk ($\mathcal{F}_{\chi=1}$), i.e. the diagrams with one quark-loop boundary [13]. In this case the relation (1) was modified as follows:

$$W_{eff}(S) = N\frac{\partial \mathcal{F}_{\chi=2}(S)}{\partial S} + \mathcal{F}_{\chi=1}(S) \tag{2}$$

The validity of this approach was tested considering a theory whose lagrangian contains a mass term for the quark fields q $(\widetilde{q})$, a Yukawa interaction with the field in the adjoint ϕ and a quadratic tree-level superpotential for ϕ. This model has been studied also for different gauge groups [19, 20, 21] and a generalized Yukawa coupling [22].

In this letter we focus on a generalization of this kind of models, with a $\mathcal{N} = 1$ $U(N)$ gauge theory with N_f flavors and an *arbitrary* tree-level polynomial superpotential for ϕ, i.e.

$$W(\phi) = \sum_{k=1}^{n+1} \frac{1}{k} g_k \phi^k \tag{3}$$

We want to obtain the matter contribution $\mathcal{F}_{\chi=1}(S)$ for this general case. Following [23] we study the maximally-confining phase of the theory and compute $\mathcal{F}_{\chi=1}$ explicitly as a power series in S, i.e.

$$\mathcal{F}_{\chi=1}(S) = \sum_j P_j S^j \tag{4}$$

We find that the coefficients P_j for given j depend only on the g_k couplings in the potential with $k < 2j$.

Our general derivation naturally includes the results obtained by Gomez-Reino [24], who solves a $SU(N)$ matrix model using the properties of the factorization of the Seiberg-Witten curve. This is a direct consequence of the fact that for a maximally confining $SU(N)$ theory only the moduli carrying even indices contribute to the factorization of the Seiberg-Witten curve [25]. Therefore restricting our result to the case of an even polynomial tree-level superpotential we recover the results in [24].

We consider a $\mathcal{N}=1$ $U(N)$ supersymmetric gauge theory obtained by softly breaking $\mathcal{N}=2$ supersymmetry via the introduction of a tree-level superpotential. The action is given by

$$S = S_{matter} + S_{gauge} + S_{break} \tag{5}$$

with

$$\begin{aligned} S_{matter} &= \int d^4x \int d^2\theta \; d^2\overline{\theta} \; \left(e^{-V}\overline{\phi} e^V \phi + \overline{q} e^V q + \widetilde{q} e^V \overline{\widetilde{q}} \right) \\ S_{gauge} &= 2\pi i\tau \int d^4x \int d^2\theta \; Tr(W^\alpha W_\alpha) \\ S_{break} &= \int d^4x \int d^2\theta \; Tr\Big(W_{tree}(\phi, q, \widetilde{q}) \Big) \end{aligned} \tag{6}$$

and

$$W_{tree}(\phi, q, \widetilde{q}) = W(\phi) + m\widetilde{q}q - \widetilde{q}\phi q \tag{7}$$

where ϕ is the matter field in the adjoint representation of $U(N)$, q and $\widetilde{q}$ N_f pairs of quark fields (with mass m) in the fundamental and anti-fundamental and W^α the field-strength of the theory. The superpotential $W(\phi)$ has the general form given in (3).

At the quantum level the vacua of the theory are determined by the appearance of a gaugino condensate described by a chiral superfield

$$S = \frac{1}{32\pi^2} Tr(W^\alpha W_\alpha) \tag{8}$$

It is the effective superpotential $W_{eff}(S)$ which encodes the vacuum structure of the theory.

We follow the matrix model approach of [1, 2, 6] with the generalization to include fundamental matter fields [13], so that we replace ϕ with a $N \times N$ hermitian matrix $\Phi^a{}_b$ $(a, b = 1, ..., N)$, q with a $N \times N_f$ matrix Q^a_α and $\widetilde{q}$ with a $N_f \times N$ matrix $\widetilde{Q}^\alpha_a$ $(\alpha = 1, ..., N_f)$ and write the matrix integral

$$Z = \frac{1}{Vol(U(N))} \int d\Phi \; dQ \; d\widetilde{Q} \; e^{-\frac{1}{g_s} W_{tree}(\Phi, Q, \widetilde{Q})} \tag{9}$$

where $Vol(U(N))$ is the volume of the gauge group and g_s is the string coupling.

It is well known [13] that in the 't Hooft limit, where we let $N \to \infty$, $g_s \to 0$ while keeping Ng_s fixed, the matrix model partition function Z receives contributions from

planar diagrams both with the topology of the sphere ($\chi = 2$) and the topology of the disk ($\chi = 1$) corresponding respectively to 0 and 1 quark boundary. Thus, if we call $\mathcal{F}$ the free energy of the model, we have

$$Z = e^{\mathcal{F}} \approx e^{\frac{\mathcal{F}_{\chi=2}}{g_s^2} + \frac{\mathcal{F}_{\chi=1}}{g_s} + \dots} \tag{10}$$

Now, if we interpret $Ng_s = S$ as the glueball chiral superfield of the gauge theory (5), we are led to the expression in (2.) The first term in (2) is exactly the glueball superpotential conjectured by Dijkgraaf-Vafa for theories with fields only in the adjoint representation while the second one comes from the extension [13] to include quark fields in the (anti)fundamental.

We want to evaluate this matter contribution $\mathcal{F}_{\chi=1}$ for an arbitrary polynomial superpotential for Φ of the form (3). As shown in [11] $\mathcal{F}_{\chi=1}$ can be written in terms of an hyperelliptic curve $y(x)$ as follows:

$$\mathcal{F}_{\chi=1} = -\frac{1}{2} N_f \int_m^{\Lambda_0} \left(W'(x) - y(x)\right)\, dx \tag{11}$$

where m is the mass of the quarks and Λ_0 a regularization cut-off. In order to integrate the hyperelliptic curve $y(x)$ in (11) here we will follow the approach of [23]. We study the so-called maximally confining phase of the theory so that the hyperelliptic curve degenerates as

$$y(x) = \sqrt{W'(x)^2 - f_{n-1}(x)} = P_{n-1}(x)\sqrt{(x-\sigma)^2 - \mu^2} \tag{12}$$

where $P_{n-1}(x)$ is a polynomial of degree $n-1$ and σ is a parameter that can be shifted to 0 in a $U(N)$ theory. The crucial point is to perform the change of variable

$$x \to \frac{\mu}{2}\,(\xi + \xi^{-1}) \tag{13}$$

in order to expand $W(x)$ and $W'(x)$ in series of ξ and ξ^{-1}:

$$\begin{aligned} W(x) &= W\Big(\frac{\mu}{2}\,(\xi + \xi^{-1})\Big) = b_0 + \sum_{k=1}^{n+1} b_k(\xi^k + \xi^{-k}) \\ W'(x) &= W'\Big(\frac{\mu}{2}\,(\xi + \xi^{-1})\Big) = c_0 + \sum_{k=1}^{n} c_k(\xi^k + \xi^{-k}) \end{aligned} \tag{14}$$

In this way one obtains (see [23] for details)

$$y(x) = P_{n-1}(x)\sqrt{x^2 - \mu^2} = \sum_{k=1}^{n} c_k(\xi^k - \xi^{-k}) \tag{15}$$

which allows to solve for S

$$S \equiv \int_{-\mu}^{+\mu} y(x)\, dx = \mu \frac{c_1}{2} \tag{16}$$

and gives also

$$\int y(x)\ dx = -\mu c_1 log(\xi) + b_0 + \sum_{k=1}^{n+1} b_k\left(\xi^k - \xi^{-k}\right) \tag{17}$$

Now we need to compute the coefficients b_k and c_k in (14). Performing the change of variable (13) we obtain

$$W(x) = \sum_{j=1}^{n+1} \frac{g_j}{j}\left(\frac{\mu}{2}\right)^j \left(\begin{array}{c} j \\ \frac{j}{2} \end{array} \right) + \sum_{k=1}^{n+1}\left(\xi^k + \xi^{-k}\right)\sum_{j=k}^{n+1}\frac{g_j}{j}\left(\frac{\mu}{2}\right)^j \left(\begin{array}{c} j \\ \frac{j-k}{2} \end{array} \right) \tag{18}$$

with the following convention:

$$\left(\begin{array}{c} j \\ \frac{2m+1}{2} \end{array} \right) \equiv 0 \qquad\qquad \left(\begin{array}{c} j \\ 2m \end{array} \right) \equiv \frac{j!}{(2m)!(j-2m)!} \tag{19}$$

From (14) and (18) we learn that:

$$\begin{aligned} b_0 &= \sum_{j=1}^{n+1} \frac{g_j}{j}\left(\frac{\mu}{2}\right)^j \left(\begin{array}{c} j \\ \frac{j}{2} \end{array} \right) \\ b_k &= \sum_{j=k}^{n+1} \frac{g_j}{j}\left(\frac{\mu}{2}\right)^j \left(\begin{array}{c} j \\ \frac{j-k}{2} \end{array} \right) \end{aligned} \tag{20}$$

Following an analogous procedure we obtain the value of c_1 which is needed in (16):

$$c_1 = \sum_{j=1}^{n} g_{j+1}\left(\frac{\mu}{2}\right)^j \left(\begin{array}{c} j \\ \frac{j-1}{2} \end{array} \right) \tag{21}$$

As shown in [23] one finds

$$\begin{aligned} \mathcal{F}_{\chi=1} &= -\frac{1}{2}N_f\Bigg\{ -W(m) + \mu c_1 log\left(\frac{\xi(\Lambda_0)}{\xi(m)}\right) + b_0 \\ &+ \sum_{k=1}^{n+1} b_k\left(\frac{m}{\mu}\right)^k\left[\left(1+\sqrt{1-\left(\frac{\mu}{m}\right)^2}\right)^k - \left(1-\sqrt{1-\left(\frac{\mu}{m}\right)^2}\right)^k\right]\Bigg\} \end{aligned} \tag{22}$$

where from (13) one can see that:

$$\begin{aligned} \xi(\Lambda_0) &= \frac{\Lambda_0}{\mu}\left(1+\sqrt{1-\left(\frac{\mu}{\Lambda_0}\right)^2}\right) \approx \frac{2\Lambda_0}{\mu} \qquad \Lambda_0 \to \infty \\ \xi(m) &= \frac{m}{\mu}\left(1+\sqrt{1-\left(\frac{\mu}{m}\right)^2}\right) \end{aligned} \tag{23}$$

Using the following identity

$$(1+a)^k - (1-a)^k = 2\sum_{m=0}^{[\frac{k-1}{2}]} \begin{pmatrix} k \\ 2m+1 \end{pmatrix} a^{2m+1} \tag{24}$$

we can rewrite (22) as

$$\begin{aligned} \mathcal{F}_{\chi=1} &= -\frac{1}{2}N_f\Bigg[- W(m) + S\ log\Big(\frac{2\Lambda_0}{m}\Big) + b_0 - 2S\ log\Big(\frac{1+\sqrt{1-(\mu/m)^2}}{2}\Big) \\ &\quad + 2\sum_{k=1}^{n+1} b_k \Big(\frac{m}{\mu}\Big)^k \sum_{l=0}^{[\frac{k-1}{2}]} \begin{pmatrix} k \\ 2l+1 \end{pmatrix} \Big(1-\Big(\frac{\mu}{m}\Big)^2\Big)^{\frac{2l+1}{2}}\Bigg] \end{aligned} \tag{25}$$

This equation can be easily expanded in series of μ^2. Using

$$\begin{aligned} log\ \Big(\frac{1+\sqrt{1-(\mu/m)^2}}{2}\Big) &= -\sum_{j\geq 1}\frac{1}{2j}\frac{1}{2^{2j}}\begin{pmatrix} 2j \\ j \end{pmatrix}\Big(\frac{\mu}{m}\Big)^{2j} \\ \Big(1-\Big(\frac{\mu}{m}\Big)^2\Big)^{\frac{2l+1}{2}} &= \sum_{j\geq 0}(-)^j \begin{pmatrix} \frac{2l+1}{2} \\ j \end{pmatrix}\Big(\frac{\mu}{m}\Big)^{2j} \end{aligned} \tag{26}$$

and the expansion in (20), finally we obtain

$$\begin{aligned} \mathcal{F}_{\chi=1} &= -\frac{1}{2}N_f\Bigg[-\sum_{j=1}^{n+1}\frac{g_j}{j}m^j + S\ log\ \Big(\frac{2\Lambda_0}{m}\Big) + \sum_{j=1}^{[\frac{n+1}{2}]}\frac{g_{2j}}{2j}\Big(\frac{\mu}{2}\Big)^{2j}\begin{pmatrix} 2j \\ j \end{pmatrix} \\ &\quad + \Bigg(\sum_{j=1}^{[\frac{n+1}{2}]} g_{2j}\Big(\frac{\mu}{2}\Big)^{2j}\begin{pmatrix} 2j \\ j \end{pmatrix}\Bigg)\Bigg(\sum_{j\geq 1}\frac{1}{2j}\frac{1}{2^{2j}}\begin{pmatrix} 2j \\ j \end{pmatrix}\Big(\frac{\mu}{m}\Big)^{2j}\Bigg) \\ &\quad + 2\sum_{k=1}^{n+1}\Bigg(\sum_{i=k}^{n+1}\frac{g_i}{i}\Big(\frac{\mu}{2}\Big)^i\begin{pmatrix} i \\ \frac{i-k}{2} \end{pmatrix}\Bigg)\sum_{l=0}^{[\frac{k-1}{2}]}\begin{pmatrix} k \\ 2l+1 \end{pmatrix}\Bigg(\sum_{j\geq 0}(-)^j\begin{pmatrix} \frac{2l+1}{2} \\ j \end{pmatrix}\Big(\frac{\mu}{m}\Big)^{2j-k}\Bigg)\Bigg] \end{aligned} \tag{27}$$

Our final aim is to write $\mathcal{F}_{\chi=1}$ as a power series in S. In order to do so we have to work on (27) showing that it can be drastically simplified.

Let us consider a matrix model with N_f flavors and a general superpotential for the chiral superfield in the adjoint representation of $U(N)$, given by the sum of an even polynomial W_{2n} and an odd one W_{2n+1} :

$$W(x) = W_{2n}(x) + W_{2n+1}(x) = \sum_{j=1}^{n}\frac{g_{2j}}{2j}x^{2j} + \sum_{j=1}^{n}\frac{g_{2j+1}}{2j+1}x^{2j+1} \tag{28}$$

We notice that, as explained in [25], the case $W(x) = W_{2n}(x)$ corresponds to a $SU(N)$ theory (for which only the even terms contribute to the glueball superpotential). The $SU(N)$ theory has been considered in [24], where the effective superpotential was computed order by order in S using factorization properties of the Seiberg-Witten curve. Solving the general case in (28) we will be able to compare our results with those in [24].

First we rewrite (27) separating the even part from the odd one:

$$\begin{aligned}
\mathcal{F}_{\chi=1} = -\frac{1}{2}N_f\Bigg[& -\sum_{j=1}^{n}\frac{g_{2j}}{2j}m^{2j} - \sum_{j=1}^{n}\frac{g_{2j+1}}{2j+1}m^{2j+1} + S\ log\left(\frac{2\Lambda_0}{m}\right) + \\
& + \sum_{j=1}^{n}\frac{g_{2j}}{2j}\left(\frac{\mu}{2}\right)^{2j}\binom{2j}{j} + \left(\sum_{i=1}^{n} g_{2i}\left(\frac{\mu}{2}\right)^{2i}\binom{2i}{i}\right)\left(\sum_{j\geq 1}\frac{1}{2j}\frac{1}{2^{2j}}\binom{2j}{j}\left(\frac{\mu}{m}\right)^{2j}\right) + \\
& + 2\sum_{k=1}^{n}\left(\sum_{i=k}^{n}\frac{g_{2i}}{2i}\left(\frac{\mu}{2}\right)^{2i}\binom{2i}{i-k}\right)\sum_{l=0}^{k-1}\binom{2k}{2l+1}\left(\sum_{j\geq 0}(-)^j\binom{\frac{2l+1}{2}}{j}\left(\frac{\mu}{m}\right)^{2(j-k)}\right) + \\
& + \sum_{k=0}^{n}\left(\sum_{i=k}^{n}\frac{g_{2i+1}}{2i+1}\left(\frac{1}{2}\right)^{2i}\binom{2i+1}{i-k}\right)\sum_{l=0}^{k}\binom{2k+1}{2l+1} \\
& \left(\sum_{j\geq 0}(-)^j\binom{\frac{2l+1}{2}}{j} m^{2k-2j+1}\mu^{2(j-k+i)}\right)\Bigg] \qquad (29)
\end{aligned}$$

In order to obtain $\mathcal{F}_{\chi=1}$ as a power series of S we proceed in two steps. First we look for an expansion in μ^2. Then we will express μ^2 as a power series in S by means of the formulas (16) and (21). To this end we organize eq. (29) in the following way:

$$\begin{aligned}
\mathcal{F}_{\chi=1} = -\frac{1}{2}N_f\Bigg[& -\sum_{j=1}^{n}\frac{g_{2j}}{2j}m^{2j} - \sum_{j=1}^{n}\frac{g_{2j+1}}{2j+1}m^{2j+1} + S\ log\left(\frac{2\Lambda_0}{m}\right) + \\
& + \sum_{j=1}^{n} A_j(\mu^2)^j + \sum_{j\geq 2} B_j(\mu^2)^j + \sum_{j\geq 0} C_j(\mu^2)^j + \sum_{j\geq 0}\widetilde{C}_j(\mu^2)^j\Bigg] \qquad (30)
\end{aligned}$$

where the seven terms correspond to the seven terms in (29).
The A_j coefficients are immediately identified

$$A_j \equiv \frac{g_{2j}}{2j}\frac{1}{2^{2j}}\binom{2j}{j}. \qquad (31)$$

The B_j coefficients are not too difficult to compute and one finds

$$B_j \equiv \sum_{\substack{i=1 \\ i\leq n}}^{j-1}\frac{g_{2i}}{2^{2j}}\binom{2i}{i}\binom{2(j-i)}{j-i}\frac{1}{2(j-i)m^{2(j-i)}} \qquad (32)$$

On the other hand the computation of C_j and $\widetilde{C}_j$ is a more difficult task. Let us concentrate on them. In the last two terms in (29) we perform the change of index $i \to i+k$ and a shift $j \to j-i$. In this way we obtain

$$
\begin{aligned}
C_j &\equiv \sum_{k=1}^{n} \sum_{\substack{i=k \\ i\leq n}}^{j+k} \frac{g_{2i}}{2i} \begin{pmatrix} 2i \\ i-k \end{pmatrix} \frac{1}{2^{2i-1}} \frac{(-)^{j-i+k}}{m^{2(j-i)}} \left[\sum_{l=0}^{k-1} \begin{pmatrix} 2k \\ 2l+1 \end{pmatrix} \begin{pmatrix} \frac{2l+1}{2} \\ j-i+k \end{pmatrix} \right] \qquad (33) \\
\widetilde{C}_j &\equiv \sum_{k=0}^{n} \sum_{\substack{i=k \\ i\leq n}}^{j+k} \frac{g_{2i+1}}{2i+1} \begin{pmatrix} 2i+1 \\ i-k \end{pmatrix} \frac{(-)^{j-i+k}}{2^{2i} m^{2(j-i)-1}} \left[\sum_{l=0}^{k} \begin{pmatrix} 2k+1 \\ 2l+1 \end{pmatrix} \begin{pmatrix} \frac{2l+1}{2} \\ j-i+k \end{pmatrix} \right] \qquad (34)
\end{aligned}
$$

The sum over l in (33) and (34) can be rewritten in a nice form:

$$
\sum_{l=0}^{k-1} \begin{pmatrix} 2k \\ 2l+1 \end{pmatrix} \begin{pmatrix} \frac{2l+1}{2} \\ j-s+k \end{pmatrix} \equiv 2^{2(s-j)} \frac{k}{(j-s+k)!} \frac{\Gamma(k+s-j)}{\Gamma(2s-2j+1)} \qquad (35)
$$

$$
\sum_{l=0}^{k-1} \begin{pmatrix} 2k+1 \\ 2l+1 \end{pmatrix} \begin{pmatrix} \frac{2l+1}{2} \\ j-s+k \end{pmatrix} \equiv \frac{2^{-2k-1}\sqrt{\pi}}{(j-s+k)!(2k)!} \frac{\Gamma(2k+2)\,\Gamma(2j-2s-1)}{\Gamma(k-j+s+\frac{3}{2})\,\Gamma(2j-2s-2k-1)} \qquad (36)
$$

We start computing the lowest terms:

$$
\begin{aligned}
C_0 &= \sum_{k=1}^{n} \frac{m^{2k}}{2^{2k-1}} \frac{g_{2k}}{2k} \sum_{l=0}^{k-1} \begin{pmatrix} 2k \\ 2l+1 \end{pmatrix} = \sum_{k=1}^{n} m^{2k} \frac{g_{2k}}{2k} \\
\widetilde{C}_0 &= \sum_{k=0}^{n} \frac{m^{2k+1}}{2^{2k}} \frac{g_{2k+1}}{2k+1} \sum_{l=0}^{k} \begin{pmatrix} 2k+1 \\ 2l+1 \end{pmatrix} = \sum_{k=0}^{n} m^{2k+1} \frac{g_{2k+1}}{2k+1} \qquad (37)
\end{aligned}
$$

In fact these coefficients are exactly cancelled by the first two terms in (30). Moreover we want to show that all the other coefficients C_j and $\widetilde{C}_j$, $j \neq 0$ receive contributions only from the first $2j$ and $2j-1$ terms of the polynomial superpotential respectively.

Therefore we need to show that every term in (33) and (34) with an index $i > j$ does not contribute. Let us consider $i = s$ for some $s > j$. It is not hard to see that the contribution in (33) proportional to g_{2s} can be written as a sum over $k = s-j, ..., s$

$$
g_{2s} \left[\sum_{k=s-j}^{s} \frac{1}{s} \begin{pmatrix} 2s \\ s-k \end{pmatrix} \frac{1}{2^{2s}} \frac{(-)^{j-s+k}}{m^{2(j-s)}} \left(\sum_{l=0}^{k-1} \begin{pmatrix} 2k \\ 2l+1 \end{pmatrix} \begin{pmatrix} \frac{2l+1}{2} \\ j-s+k \end{pmatrix} \right) \right] \qquad (38)
$$

In the same way the contribution in (34) proportional to g_{2s+1} becomes

$$
g_{2s+1} \left[\sum_{k=s-j}^{s} \frac{1}{2s+1} \begin{pmatrix} 2s+1 \\ s-k \end{pmatrix} \frac{1}{2^{2s}} \frac{(-)^{j-s+k}}{m^{2(j-s)-1}} \left(\sum_{l=0}^{k} \begin{pmatrix} 2k+1 \\ 2l+1 \end{pmatrix} \begin{pmatrix} \frac{2l+1}{2} \\ j-s+k \end{pmatrix} \right) \right] \qquad (39)
$$

Now, setting $t \equiv s - j$, $t > 0$ (38) becomes proportional to

$$\sum_{k=0}^{j} \left(\begin{array}{c} 2(j+t) \\ j-k \end{array} \right) (-)^k (k+t) \, \frac{\Gamma(k+2t)}{\Gamma(k+1)} =$$
$$= \frac{4^j \, t \, \Gamma(j+1/2) \, \Gamma(2t)}{\sqrt{\pi} \, j \, \Gamma(j)} - \frac{\Gamma(2j+1) \, \Gamma(2t+1)}{2 \, j^2 \, \Gamma^2(j)} =$$
$$= \frac{2 \, j \, t \, \Gamma(2t)}{\sqrt{\pi} \, j^2 \, \Gamma^2(j)} \Big(2^{2j-1} \, \Gamma(j) \, \Gamma(j+1/2) - \sqrt{\pi} \, \Gamma(2j) \Big) \equiv 0 \tag{40}$$

In a similar manner from (39) we obtain

$$\sum_{k=s-j}^{s} \frac{(-)^k}{2^{2k}} \frac{\Gamma(2k+2)}{(j-s+k)! \, \Gamma(k-j+s+\frac{3}{2})} \, \frac{\Gamma(2j-2s-1)}{\Gamma(2j-2s-2k-1)(2k)!} \tag{41}$$

which is proportional to

$$4^j \, \Gamma(j+\frac{1}{2}) \, \Gamma^2(j+1) - 2\Gamma(j) \Big[j \, \sqrt{\pi} \, (j-2s-2) \, \Gamma(2j) + 4^j (1+s) \, \Gamma(j+\frac{1}{2}) \, \Gamma(j+1) \Big] \equiv 0 \tag{42}$$

We have used repeatedly the following property of the Γ matrices (for integer j):

$$\Gamma(j+1/2) \equiv \frac{\sqrt{\pi}}{2^{2j-1}} \, \frac{\Gamma(2j)}{\Gamma(j)} \tag{43}$$

This completes our proof.

At this point we are left with:

$$C_j = \sum_{s=1}^{j} c_{js} g_{2s} \qquad\qquad \widetilde{C}_j = \sum_{s=1}^{j-1} \widetilde{c}_{js} g_{2s+1}$$

where

$$c_{js} \equiv \sum_{k=1}^{s} \frac{(-)^{j-s+k}}{2^{2s} s \, m^{2(j-s)}} \left(\begin{array}{c} 2s \\ s-k \end{array} \right) \sum_{l=0}^{k-1} \left(\begin{array}{c} 2k \\ 2l+1 \end{array} \right) \left(\begin{array}{c} \frac{2l+1}{2} \\ j-s+k \end{array} \right) \tag{44}$$

and

$$\widetilde{c}_{js} \equiv \sum_{k=1}^{s} \frac{(-)^{j-s+k}}{2^{2s}(2s+1) m^{2(j-s)}} \left(\begin{array}{c} 2s+1 \\ s-k \end{array} \right) \sum_{l=0}^{k} \left(\begin{array}{c} 2k+1 \\ 2l+1 \end{array} \right) \left(\begin{array}{c} \frac{2l+1}{2} \\ j-s+k \end{array} \right) \tag{45}$$

Using (35) and (36) we find:

$$c_{jj} = -\frac{1}{2^{2j} 2j} \left(\begin{array}{c} 2j \\ j \end{array} \right)$$
$$c_{js} = -\frac{1}{2^{2j-1} j \, s \, m^{2(j-s)}} \, \frac{\Gamma(2s)}{\Gamma^2(s)} \left(\begin{array}{c} 2(j-s)-1 \\ j-s \end{array} \right) \qquad s < j$$
$$\widetilde{c}_{js} = -\frac{1}{2^{2j-1} j \, s^2 \, m^{2(j-s)-1}} \, \frac{\Gamma(2s+1)}{\Gamma^2(s)} \left(\begin{array}{c} 2(j-s)-2 \\ j-s-1 \end{array} \right) \qquad s < j \tag{46}$$

Note that the A_j terms are exactly cancelled by c_{jj} for $j = 1, ...n$ computed in (46), leaving no term linear in μ^2 (then no term linear in S except for the standard piece $S\ log(2\Lambda_0/m)$). So we can rewrite (30) as follows:

$$\mathcal{F}_{\chi=1} = -\frac{1}{2}N_f\left[S\ log\left(\frac{2\Lambda_0}{m}\right) + \sum_{j\geq 2}(B_j + C_j^n + \widetilde{C}_j)(\mu^2)^j\right] \tag{47}$$

where we have defined

$$C_j^n \equiv \begin{cases} C_j - g_{2j}c_{jj} & j \leq n \\ C_j & j > n \end{cases} \tag{48}$$

We give explicitly the form of the lowest terms for the coefficients B_j:

$$\begin{aligned} B_2 &= +\frac{1}{8}\frac{g_2}{m^2} \\ B_3 &= +\frac{3}{64}\frac{g_2}{m^4} + \frac{3}{32}\frac{g_4}{m^2} \\ B_4 &= +\frac{5}{192}\frac{g_2}{m^6} + \frac{9}{256}\frac{g_4}{m^4} + \frac{5}{64}\frac{g_6}{m^2} \end{aligned} \tag{49}$$

and for the coefficients C_j and $\widetilde{C}_j$:

$$\begin{aligned} C_1 &= -\frac{g_2}{4} \\ C_2 &= -\frac{g_2}{16m^2} - \frac{3}{32}g_4 \\ C_3 &= -\frac{3}{96}\frac{g_2}{m^4} - \frac{3}{96}\frac{g_4}{m^2} - \frac{5}{96}g_6 \\ C_4 &= -\frac{5}{256}\frac{g_2}{m^6} - \frac{9}{512}\frac{g_4}{m^4} - \frac{5}{256}\frac{g_6}{m^2} - \frac{35}{1024}g_8 \end{aligned} \tag{50}$$

and

$$\begin{aligned} \widetilde{C}_1 &= 0 \\ \widetilde{C}_2 &= -\frac{g_3}{8m} \\ \widetilde{C}_3 &= -\frac{g_3}{24m^3} - \frac{g_5}{16m} \\ \widetilde{C}_4 &= -\frac{3}{128}\frac{g_3}{m^5} - \frac{3}{128}\frac{g_5}{m^3} - \frac{5}{128}\frac{g_7}{m} \end{aligned} \tag{51}$$

As a final step we want to reexpress $\mathcal{F}_{\chi=1}$ in a power series of S. From (16) and (21) we know that

$$S = \frac{1}{2}\sum_{j=1}^{[\frac{n+1}{2}]} g_{2j}\left(\frac{\mu}{2}\right)^{2j}\begin{pmatrix} 2j \\ j \end{pmatrix} \tag{52}$$

We want to invert this relation in order to obtain μ^2 in terms of S. We look for an expression of the form

$$\mu^2 = \sum_{m>0} a_m S^m \tag{53}$$

where a_m are the coefficients to be determined. Inserting (53) into (52) we obtain

$$\begin{aligned} S &= \frac{1}{2}\sum_{j=1}^{[\frac{n+1}{2}]} \frac{g_{2j}}{2^{2j}} \begin{pmatrix} 2j \\ j \end{pmatrix} \Big(\sum_{m>0} a_m S^m\Big)^j \\ &= \frac{1}{2}\sum_{j=1}^{[\frac{n+1}{2}]} \frac{g_{2j}}{2^{2j}} \begin{pmatrix} 2j \\ j \end{pmatrix} \sum_{\substack{p_1,p_2,...=0 \\ p_1+p_2+...=j}}^{j} M(j;p_1,p_2,...) a_1^{p_1} a_2^{p_2} ... S^{p_1+2p_2+3p_3+...} \end{aligned} \tag{54}$$

where we have defined the coefficient:

$$M(j;p_1,p_2,...,p_t,...) \equiv \frac{j!}{p_1! p_2! ... p_t! ...} \tag{55}$$

With a bit of labor one can argue that the general coefficient a_m can be expressed recursively in terms of $a_1, a_2, ..., a_{m-1}$:

$$\begin{aligned} a_1 &= \frac{4}{g_2} \\ a_m &= -\sum_{\substack{p_1,...,p_{m-1}=0 \\ p_1+2p_2+...+(m-1)p_{m-1}=m \\ p_1+p_2+...+p_{m-1}\leq[\frac{n+1}{2}]}}^{m} \frac{2}{g_2} \begin{pmatrix} 2(p_1+...+p_{m-1}) \\ p_1+...+p_{m-1} \end{pmatrix} a_1^{p_1} ... a_{m-1}^{p_{m-1}} \cdot \\ &\quad \cdot \frac{(p_1+...+p_{m-1})!}{(p_1)!...(p_{m-1})!} \frac{g_{2(p_1+...+p_{m-1})}}{2^{2(p_1+...+p_{m-1})}} \end{aligned} \tag{56}$$

Here we give the expressions of the first coefficients:

$$\begin{aligned} a_2 &= -12\frac{g_4}{g_2^3} \\ a_3 &= +72\frac{g_4^2}{g_2^5} - 40\frac{g_6}{g_2^4} \\ a_4 &= -540\frac{g_4^3}{g_2^7} + 600\frac{g_4 g_6}{g_2^6} - 140\frac{g_8}{g_2^5} \\ a_5 &= +4536\frac{g_4^4}{g_2^9} - 7560\frac{g_4^2 g_6}{g_2^8} + 2520\frac{g_4 g_8}{g_2^7} + 1200\frac{g_6^2}{g_2^7} - 504\frac{g_{10}}{g_2^6} \end{aligned} \tag{57}$$

Now, using (53) and (56) we are able to give the expansion of $\mathcal{F}_{\chi=1}$ in series of S:

$$\mathcal{F}_{\chi=1} = -\frac{1}{2}N_f\left[S\,log\left(\frac{2\Lambda_0}{m}\right) + \sum_{k\geq 2}\left(\sum_{j=2}^{k}(B_j + C_j^n + \widetilde{C}_j) \sum_{\substack{p_1...p_k=0 \\ p_1+...+p_k=j \\ p_1+...+kp_k=k}}^{j} M(j;p_1,...,p_k)\, a_1^{p_1} ... a_k^{p_k}\right) S^k\right]$$

$$
\begin{aligned}
= \quad & -N_f \Bigg[\frac{1}{2} S \; log \; \Big(\frac{2\Lambda_0}{m}\Big) + \Big(\frac{1}{2g_2 m^2} - \frac{g_3}{g_2^2 m}\Big) S^2 \\
& + \Big(\frac{1}{2g_2^2 m^4} - \frac{4}{3}\frac{g_3}{g_2^3 m^3} - \frac{g_4}{g_2^3 m^2} + 6\frac{g_3 g_4}{g_2^4 m} - 2\frac{g_5}{g_2^3 m}\Big) S^3 \\
& + \Big(\frac{5}{6}\frac{1}{g_2^3 m^6} - 3\frac{g_3}{g_2^4 m^5} + \frac{9}{2}\frac{g_4^2}{g_2^5 m^2} - \frac{9}{4}\frac{g_4}{g_2^4 m^4} + 12\frac{g_3 g_4}{g_2^5 m^3} - 45\frac{g_3 g_4^2}{g_2^6 m} \\
& + 18\frac{g_4 g_5}{g_2^5 m} - 3\frac{g_5}{g_2^4 m^3} - \frac{5}{2}\frac{g_6}{g_2^4 m^2} + 20\frac{g_3 g_6}{g_2^5 m} - 5\frac{g_7}{g_2^4 m}\Big) S^4 + ... \Bigg] \qquad (58)
\end{aligned}
$$

If we extract from (58) the even contributions we reproduce exactly what has been computed in [24].

In order to have the full expression of the glueball superpotential (2) we have to add the $\chi = 2$ contribution to (58). One can find the implicit solution to this problem in [25] where the $\chi = 2$ contribution is given in terms of some parameters which are non-linear functions of S. After some algebra we found that the explicit solution can be written as

$$
N \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} = -NS \, log\Big(\frac{S}{g_2 \Lambda_0^2}\Big) - N \sum_{j>0} (W_j - \widetilde{W}_j) S^j \qquad (59)
$$

where

$$
W_j \equiv \sum_{m=1}^{j-1} \frac{(-)^{m+1}}{m D_1^m} \sum_{\substack{p_1,...,p_{j-1}=0 \\ p_1+...+p_{j-1}=m \\ p_1+...+(j-1)p_{j-1}=j-1}}^{m} M(m; p_1, ..., p_{j-1}) D_2^{p_1} ... D_j^{p_{j-1}} \qquad (60)
$$

and

$$
\begin{aligned}
\widetilde{W}_j \equiv \sum_{p=2}^{2j} \frac{g_p}{p} \sum_{q=0}^{[p/2]} \binom{p}{2q} \binom{2q}{q} & \sum_{\substack{p_1,...,p_{j-1}=0 \\ p_1+...+p_{j-1}=p-2q}}^{p-2q} M(p-2q; p_1, ..., p_{j-1}) z_1^{p_1} ... z_{j-1}^{p_{j-1}} \\
& \sum_{\substack{q_1,...,q_j=0 \\ q_1+...+q_j=q \\ (p_1+q_1)+...+(j-1)(p_{j-1}+q_{j-1})+jq_j=j}}^{q} M(q; q_1, ..., q_j) D_1^{q_1} ... D_j^{q_j} \qquad (61)
\end{aligned}
$$

are defined in terms of some parameters D_j and z_j which can be computed recursively

$$
D_j \equiv \sum_{k>0} d_k \sum_{\substack{p_1,...,p_j=0 \\ p_1+...+p_j=k \\ p_1+...+jp_j=j}}^{k} M(k; p_1, ..., p_j) z_1^{p_1} ... z_j^{p_j} \qquad (62)
$$

and

$$z_1 = -2\frac{g_3}{g_2^2}$$

$$z_j \equiv -\frac{1}{g_2 d_1}\sum_{p=2}^{2j}\frac{g_p}{p}\sum_{q=1}^{[p/2]} q\begin{pmatrix} p \\ 2q \end{pmatrix}\begin{pmatrix} 2q \\ q \end{pmatrix}\sum_{\substack{p_1,\ldots,p_j=0 \\ p_1+\ldots+p_j=q}}^{q} M(q;p_1,\ldots,p_j)d_1^{p_1}\ldots d_j^{p_j}$$

$$\sum_{\substack{q_1,\ldots,q_{j-1}=0 \\ q_1+\ldots+q_{j-1}=p-2q+p_1+\ldots+jp_j \\ q_1+\ldots+(j-1)q_{j-1}=j}}^{p-2q+p_1+\ldots+jp_j} M(p-2q+p_1+\ldots+jp_j;q_1,\ldots,q_{j-1})z_1^{q_1}\ldots z_{j-1}^{q_{j-1}} \tag{63}$$

The d_j coefficients which appear in (69) and (70) are also given recursively:

$$d_1 = -\frac{g_2}{2g_3}$$

$$d_j \equiv -\frac{1}{2g_3}\sum_{p=3}^{2j} g_{p+1}\sum_{q=0}^{[p/2]}\begin{pmatrix} p \\ 2q \end{pmatrix}\begin{pmatrix} 2q \\ q \end{pmatrix}\sum_{\substack{p_1,\ldots,p_{j-1}=0 \\ p_1+\ldots+p_{j-1}=q \\ p-2q+p_1+\ldots+(j-1)p_{j-1}=j}}^{q} M(q;p_1,\ldots,p_{j-1})\ d_1^{p_1}\ldots d_{j-1}^{p_{j-1}} \tag{64}$$

Explicitly to the fourth order in S we obtain

$$\begin{aligned} N\frac{\partial\mathcal{F}_{\chi=2}}{\partial S} &= -NS\Big(log\Big(\frac{S}{g_2\Lambda_0^2}\Big)-1\Big) - N\Bigg[\Big(2\frac{g_3^2}{g_2^3}-\frac{3}{2}\frac{g_4}{g_2^2}\Big)S^2 + \Big(\frac{32}{3}\frac{g_3^4}{g_2^6}-24\frac{g_3^2g_4}{g_2^5}+ \\ &+\frac{9}{2}\frac{g_4^2}{g_2^4}+12\frac{g_3g_5}{g_2^4}-\frac{10}{3}\frac{g_6}{g_2^3}\Big)S^3+\Big(\frac{280}{3}\frac{g_3^6}{g_2^9}-340\frac{g_3^4g_4}{g_2^8}+270\frac{g_3^2g_4^2}{g_2^7}-\frac{45}{2}\frac{g_4^3}{g_2^6}+ \\ &+200\frac{g_3^3g_5}{g_2^7}-180\frac{g_3g_4g_5}{g_2^6}+18\frac{g_5^2}{g_2^5}-100\frac{g_3^2g_6}{g_2^6}+30\frac{g_4g_6}{g_2^5}+40\frac{g_3g_7}{g_2^5}-\frac{35}{4}\frac{g_8}{g_2^4}\Big)S^4+\ldots\Bigg] \end{aligned} \tag{65}$$

Let us notice that the S^k term in this expression depends only on the low order couplings $g_3, g_4, \ldots, g_{2k}$, in a way parallel to what we have found for the case of fundamental matter (see (58)). When only an adjoint matter field is present this behavior follows immediately from the perturbative analysis in [9]. There it was found that contributions to the S^k term come from a planar diagram with exactly k loops, which can contain vertices with at most $2k$ external legs.
What we found in (58) is the corresponding situation where now the graphs have an extra loop of fundamental matter. It is very easy to write down the graphs corresponding to every single term in the expressions (58) and (65) since the factors of g_2 and m in the denominators count the number of adjoint and fundamental propagators respectively.

To summarize, we have computed the effective glueball superpotential for a $\mathcal{N}=1$ $U(N)$ gauge theory with matter in the fundamental (N_f flavors) and a field in the adjoint with a general polynomial tree-level superpotential. This has been done by using the technology developed in [23] for computing the contribution to the matrix model partition function coming from diagrams with the disk topology. The full contribution to the glueball superpotential is given in the expressions (58) and (59).
Our results generalize the ones obtained in [24] since as discussed in [25] the $SU(N)$ theory considered there is equivalent to a $U(N)$ model with an even polynomial superpotential for the adjoint field.
It would be interesting to extend these results to the case of a superpotential admitting several vacua.

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